

FROBENIUS NUMBERS OF PYTHAGOREAN TRIPLES

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ABSTRACT. Given relatively prime integers a_1, \dots, a_n , the Frobenius number $g(a_1, \dots, a_n)$ is defined as the largest integer which cannot be expressed as $x_1a_1 + \dots + x_na_n$ with x_i nonnegative integers.

In this article, we give the Frobenius number of primitive Pythagorean triples. That is,

$$g(m^2 - n^2, 2mn, m^2 + n^2) = (m - 1)(m^2 - n^2) + (m - 1)(2mn) - (m^2 + n^2).$$

1. INTRODUCTION

The history of Frobenius numbers goes back to J. J. Sylvester. He proposed a problem to find the largest number for two relatively prime integers a and b which cannot be expressed as $ax + by$ with x and y nonnegative integers [6]. His answer was $ab - a - b$.

F. G. Frobenius emphasized this problem and asked to study such a number for more than two integers. That is, given relatively prime integers a_1, a_2, \dots, a_n , he asked to find the largest integer $N = g(a_1, a_2, \dots, a_n)$ for which $a_1x_1 + a_2x_2 + \dots + a_nx_n = N$ has no nonnegative integer solutions. The number henceforth has been called *Frobenius number*.

Many mathematicians have studied Frobenius numbers and found some algorithms to compute them, nevertheless no one found a general exact formula. In actual, unlike Sylvester's elegant formula, there are no polynomial solutions for more than two integers [1]. In general, it is NP-hard to find the Frobenius numbers for the number of given integers [4].

So, lots of the studies have focused on specific sets of integers. For example, formulas for arithmetic progressions and geometric progressions were found [5], [3]. The formula for Fibonacci numbers F_i, F_{i+2}, F_{i+k} was found [2].

In the present article, we give a formula for primitive Pythagorean triples. This research was performed as a program of Institute of the Gifted Education in Science of Kyungnam university.

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2. MAIN RESULT

We have that $(m^2 - n^2, 2mn, m^2 + n^2)$ is a primitive Pythagorean triple if two integers $m > n$ are relatively prime and have opposite parity.

Theorem. *The Frobenius number for a Pythagorean triple is*

$$\begin{aligned} g(m^2 - n^2, 2mn, m^2 + n^2) \\ = (m - 1)(m^2 - n^2) + (m - 1)(2mn) - (m^2 + n^2). \end{aligned}$$

For convenience, let

$$\begin{aligned} A &= (m - 1)(m^2 - n^2) + (m - 1)(2mn) - (m^2 + n^2) \\ &= m(m^2 + 2mn - n^2 - 2m - 2n). \end{aligned}$$

First, we prove that $g(m^2 - n^2, 2mn, m^2 + n^2) \geq A$. To do this, we show that A cannot be expressed by $m^2 - n^2$, $2mn$, and $m^2 + n^2$.

Suppose that there exist nonnegative integers x , y , and z satisfying

$$(*) \quad x(m^2 - n^2) + y(2mn) + z(m^2 + n^2) = A.$$

Dividing both sides by m , we obtain that $(-x + z)n^2 \equiv 0 \pmod{m}$. Since m and n are relatively prime, we can set $z = x + mk$. Plug it into the above expression and divide by m . Then

$$\begin{aligned} km^2 + 2xm + 2yn + kn^2 &= m^2 + 2mn - n^2 - 2m - 2n \\ k(m^2 + n^2) + 2(xm + yn) &= (m^2 - n^2) + 2(mn - m - n) \end{aligned}$$

and we can conclude that k is odd since $m^2 + n^2$ and $m^2 - n^2$ are both odd.

Setting $k = 2\ell + 1$, we obtain

$$\ell m^2 + \ell n^2 + xm + yn + n^2 = mn - m - n.$$

If $\ell \geq 0$, then

$$(\ell m + x)m + (\ell n + y + n)n = mn - m - n,$$

but this is absurd because $g(m, n) = mn - m - n$ by Sylvester. It follows that $\ell \leq -1$. Set $\ell' = -\ell - 1 \geq 0$.

Changing roles of x and z , plugging $x = z - mk$ into the above expression $(*)$ yields

$$-mk(m^2 - n^2) + y(2mn) + z(2m^2) = m(m^2 + 2mn - n^2 - 2m - 2n).$$

Divide both sides by m and rearrange.

$$-k(m^2 - n^2) + y(2n) + z(2m) = m^2 + 2mn - n^2 - 2m - 2n$$

Using $k = 2\ell + 1 = 2(-\ell' - 1) + 1 = -2\ell' - 1$, we can write

$$\begin{aligned} \ell'(m + n)(m - n) + yn + zm &= mn - m - n \\ (\ell'(m - n) + z)m + (\ell'(m - n) + y)n &= mn - m - n. \end{aligned}$$

This is also absurd. So it was proved that

$$\begin{aligned} g(m^2 - n^2, 2mn, m^2 + n^2) \\ \geq (m-1)(m^2 - n^2) + (m-1)(2mn) - (m^2 + n^2). \end{aligned}$$

Now, we prove that $g(m^2 - n^2, 2mn, m^2 + n^2) \leq A$. That is, we show that every integer greater than A is expressed as $x(m^2 - n^2) + y(2mn) + z(m^2 + n^2)$ with $x, y, z \geq 0$. To do this, we need a lemma.

Lemma. *For a fixed positive integer b , we define y to be the smallest positive integer such that the interval $\left[\frac{b+yn}{m}, \frac{ym}{n}\right]$ contains an integer, and we also define x to be the smallest integer contained. Then, the inequality*

$$(ym + xn)(m^2 - n^2) \leq A + (m^2 - n^2) + b(2mn)$$

holds.

Proof. The existence of y is guaranteed from $\frac{n}{m} < 1 < \frac{m}{n}$.

From the condition, if $y = 0$, then x and b also vanish. Thus the inequality holds for $y = 0$ trivially.

Let us consider when $y \geq 1$. We have that $x - 1 < \frac{b+yn}{m}$ from the minimality of x . Also,

$$\frac{b + (y-1)n}{m} = \frac{b + yn}{m} - \frac{n}{m} > \frac{b + yn}{m} - 1 > x - 2.$$

We divide two cases according to comparison of $x - 1$ and $\frac{b+(y-1)n}{m}$.

Case 1: $x - 1 < \frac{b+(y-1)n}{m} < x$.

Note that

$$(1-1) \quad (x-1)m + 1 = xm - m + 1 \leq b + (y-1)n = b + yn - n.$$

If $x \leq \frac{(y-1)m}{n}$, then $\frac{b+(y-1)n}{m} < x \leq \frac{(y-1)m}{n}$ and it contradicts the minimality of y . Thus, $\frac{(y-1)m}{n} < x$ or

$$(1-2) \quad (y-1)m + 1 = ym - m + 1 \leq xn.$$

Combining the above two inequalities (1-1) and (1-2), we obtain

$$m(xm - m + 1 + n - b) \leq ymn \leq n(xn + m - 1).$$

This inequality yields a new bound for x . That is,

$$(1-3) \quad x \leq \frac{m^2 + bm - m - n}{m^2 - n^2}.$$

Now, we can conclude that

$$\begin{aligned}
& (ym + xn)(m^2 - n^2) \\
& \leq (xn + m - 1 + xn)(m^2 - n^2) && \text{by (1-2)} \\
& \leq \left(2n \times \frac{m^2 + bm - m - n}{m^2 - n^2} + m - 1 \right) (m^2 - n^2) && \text{by (1-3)} \\
& = b(2mn) + m(m^2 - n^2) + (m - 1)(2mn) - (m^2 + n^2) \\
& = A + (m^2 - n^2) + b(2mn).
\end{aligned}$$

Case 2: $x - 2 < \frac{b+(y-1)n}{m} < x - 1$.

We already verified that $x - 1 < \frac{b+yn}{m}$ at the start of the proof. Thus,

$$(2-1) \quad xm - m + 1 \leq b + yn.$$

If $x - 1 \leq \frac{(y-1)m}{n}$, then $\frac{b+(y-1)n}{m} \leq x - 1 \leq \frac{(y-1)m}{n}$ and it cotradict the minimality of y . Thus $\frac{(y-1)m}{n} < x - 1$ or

$$(2-2) \quad (y - 1)m + 1 = ym - m + 1 \leq (x - 1)n = xn - n.$$

From the above two inequalities (2-1) and (2-2),

$$m(xm - m + 1 - b) \leq ymn \leq n(xn - n + m - 1)$$

and thus

$$(2-3) \quad x \leq 1 + \frac{mn + bm - m - n}{m^2 - n^2}.$$

Then,

$$\begin{aligned}
& (ym + xn)(m^2 - n^2) \\
& \leq (xn + m - n - 1 + xn)(m^2 - n^2) && \text{by (2-2)} \\
& \leq \left(2n \times \frac{mn + bm - m - n}{m^2 - n^2} + m + n - 1 \right) (m^2 - n^2) && \text{by (2-3)} \\
& = b(2mn) + m(m^2 - n^2) + (m - 1)(2mn) - (m^2 + n^2) \\
& = A + (m^2 - n^2) + b(2mn).
\end{aligned}$$

Hence, the proof of Lemma was completed. \square

Let us resume the proof of Theorem. Let k be an arbitrary positive integer. We can choose two positive integers a and b such that

$$a(m^2 - n^2) - b(2mn) = A + k.$$

If a is chosen to be negative, consider

$$(a + 2mnt)(m^2 - n^2) - (b + (m^2 - n^2)t)(2mn) = A + k$$

for sufficiently large t .

Recall x and y in Lemma. From $\frac{b+yn}{m} \leq x \leq \frac{ym}{n}$,

$$-b + xm - yn \geq 0 \quad \text{and} \quad ym - xn \geq 0.$$

If $a + 1 \leq ym + xn$, then

$$\begin{aligned} 1 \leq k &= a(m^2 - n^2) - b(2mn) - A \\ &\leq (ym + xn - 1)(m^2 - n^2) - b(2mn) - A \\ &= (ym + xn)(m^2 - n^2) - (m^2 - n^2) - b(2mn) - A \\ &\leq A + (m^2 - n^2) + b(2mn) - (m^2 - n^2) - b(2mn) - A && \text{by Lemma} \\ &= 0. \end{aligned}$$

This contradiction yields $ym + xn \leq a$.

Therefore,

$$\begin{aligned} A + k &= a(m^2 - n^2) + (-b)(2mn) \\ &= (a - ym - xn)(m^2 - n^2) \\ &\quad + (-b + xm - yn)(2mn) \\ &\quad + (ym - xn)(m^2 + n^2) \end{aligned}$$

and $a - ym - xn \geq 0$, $-b + xm - yn \geq 0$, $ym - xn \geq 0$.

We conclude that $g(m^2 - n^2, 2mn, m^2 + n^2) = A$.

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